Mathematical Topics in Fluid Mechanics

Goals, Motivation and Outline. The objective of this course is to introduce the students to the mathematical theory of the Navier-Stokes initial-boundary value problem and to some of the solved and unsolved fundamental questions.

As is well-known, this theory presents, to date, a number of unresolved basic questions, mostly, for the physically relevant case of three-dimentional (3D) motions. Among them, certainly, the most famous is that of proving or disproving existence of 3D regular solutions for all times, for data of arbitrary "size". It should be observed that the analogous question in the two-dimensional (2D) case received a positive answer almost half a century ago.

In this course, after deriving the Navier-Stokes equations from the general continuum theory, we shall prove the main known results of existence, uniqueness and regularity of solutions to the corresponding initial-boundary value problem. Moreover, we shall present a number of significant open questions and explain why the current mathematical approaches fail to answer them. In some cases, we shall also point out possible strategies of resolution.

More specifically, the course will cover the following topics.

- Derivation of the Navier-Stokes equations and their physical relevance;
- Formulation of the initial-boundary value problem in an arbitrary domain of $\mathbb{R}^n$, $n = 2, 3$.
- Considerations on the peculiar structure of the Navier-Stokes equations.
- *A priori* estimates: The \Energy Equality" and fundamental differences between the 2D and 3D cases.
- Weak Solutions: Definition and basic properties.
- Weak solutions: Existence and Leray-Hopf class.
- Weak solutions: Uniqueness criteria.
- Weak solutions: Regularity criteria.
- Existence and uniqueness of global regular solutions in 2D.
- Classical solutions in 3D: Local and global existence results.

- Leray's \Structure Theorem": Behavior of 3D solutions at possible times of irregularity and associated fractal dimension property.

Prerequisites. The only prerequisite for this course is a basic knowledge of Hilbert, Lebesgue and Sobolev spaces.

Suggested Readings.

G.P. Galdi, An Introduction to the Navier-Stokes Initial-Boundary Value Problem, in *Fundamental Directions in Mathematical Fluid Mechanics*, G.P. Galdi, J.G. Heywood and R. Rannacher Eds, Birkhauser, 2000, 1{70. (Students will be provided with a pdf file of this article.)

G.P. Galdi, The Navier-Stokes Equations: A Mathematical Analysis, Meyers R.A. (eds) Encyclopedia of Complexity and Systems Science. Springer, New York, NY. 2009, 5925{5959 (Students will be provided with a pdf file of this article.)

J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* **63**, 1934, 193{248 (At their request, students will be provided with an English translation of this paper.)

E. Hopf, Uber die Anfganswertaufgabe fur die Hydrodynamischen Grundgleichungen, *Math. Nachr.*, **4**, 1950/1951, 213-231 (At their request, students will be provided with an English translation of this paper.)