Homeomorphisms in Sobolev spaces: Jacobian, Lusin (N) and Approximation

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This course is devoted to the analysis and geometry of *homeomorphisms in Sobolev spaces*, with a focus on the subtle interplay between weak differentiability, topology, and orientation. Once we move from smooth diffeomorphisms to Sobolev homeomorphisms (for instance $f \in W^{1,p}_{\mathrm{loc}}$), several classical geometric facts become delicate: the Jacobian determinant may only be defined almost everywhere, the change-of-variables principle may require additional structural hypotheses, and even the very meaning of "preserving orientation" must be formulated carefully in weak settings.

A central question is whether a Sobolev homeomorphism$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ can have a Jacobian determinant J_f that *changes sign on a set of positive measure*. This problem is deeply connected to the analytic formulation of orientation, to the stability of the topological degree under weak limits, and to the possibility (or impossibility) of approximating Sobolev homeomorphisms by smooth maps. Understanding when the sign of J_f is essentially constant is not merely a technical curiosity: it underpins common assumptions in geometric function theory and appears naturally in nonlinear elasticity, where orientation preservation is often encoded by constraints such as $J_f \geq 0$ almost everywhere.

The course develops the toolkit needed to address these issues. We will discuss the distinction between the *pointwise Jacobian* $\det Df$ (defined at points of approximate differentiability) and the *distributional determinant* $\mathrm{Det}\, Df$, why they may fail to coincide under minimal assumptions, and which additional integrability or structural conditions restore the expected identities.

A second major thread concerns the *Lusin property (N)* and related "size-control" properties for images of null sets. For Sobolev mappings, the implication $|E| = 0 \implies |f(E)| = 0$ is not automatic, yet it is intimately tied to change-of-variables and area-type formulas. We will see how condition (N) interacts with the Sobolev exponent, with the notion of *finite distortion*, and with borderline Orlicz–Sobolev frameworks that arise when one pushes approximation and regularity results to (or below) classical thresholds.

A third pillar is *approximation*: when and how a Sobolev homeomorphism can be approximated by diffeomorphisms (or smooth homeomorphisms) in the same Sobolev topology. We will highlight both positive approximation theorems and obstructions, including the special role of the planar case, the appearance of *bi-Sobolev* assumptions (simultaneous control of f and f^{-1}), and the role of Orlicz growth conditions in sharp approximation statements.

Prerequisites

Measure theory (Lebesgue integration), basic functional analysis, Sobolev spaces $W^{1,p}$.

Learning outcomes

At the end of the course, students should be able to:

1. Explain why orientation and Jacobian-sign information are nontrivial for Sobolev homeomorphisms, and how degree theory enters.
2. Distinguish $\det Df$ from $\operatorname{Det} Df$, and state conditions ensuring $\operatorname{Det} Df = \det Df$.
3. Describe constructions where J_f vanishes on large sets (including almost everywhere), and understand what this implies for area/change-of-variables phenomena.
4. Relate the Lusin property (N) to Jacobian integrability and to finite distortion assumptions.
5. Summarize approximation results (especially in the plane) and explain known obstructions in higher dimensions.

References

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